# MATH 2050C Lecture on 3/6/2020

[ Reminder: PS5 due today, PS6 posted. ]

**Question:** Given a seq. $(x_n)$, can we find conditions "$(\star)$" s.t.

(1) $(\star) \Rightarrow (x_n)$ convergent ?

(2) $(\star)' \Rightarrow (x_n)$ divergent ?

**Some examples:** (Divergence criteria)

$(\star)'$ : $(x_n)$ unbdd

**Some examples:** (Convergence criteria)

$(\star)$ : Squeeze Thm, or Ratio test, or limit theorems ......

$(\star)$ : $(x_n)$ bdd & "monotone"

**Def$^{\underline{n}}$:** $(x_n)$ is monotone if it is

either (i) increasing, i.e. $x_1 \leq x_2 \leq x_3 \leq \cdots$ ( $x_n \leq x_{n+1}$ $\forall n \in \mathbb{N}$ )

or (ii) decreasing, i.e. $x_1 \geq x_2 \geq x_3 \geq \cdots$ ( $x_n \geq x_{n+1}$ $\forall n \in \mathbb{N}$ )

**Note:** If inequalities above are strict, then we say that it is strictly monotone / increasing / decreasing.

**E.g.)** $(x_n) = (n)$ strictly increasing unbdd divergent

**E.g.)** $(x_n) = (\frac{1}{n})$ strictly decreasing bdd convergent

**Monotone Convergence Thm (MCT):** $(x_n)$ monotone & $\underset{\text{necessary}}{\text{bdd}} \Rightarrow (x_n)$ convergent

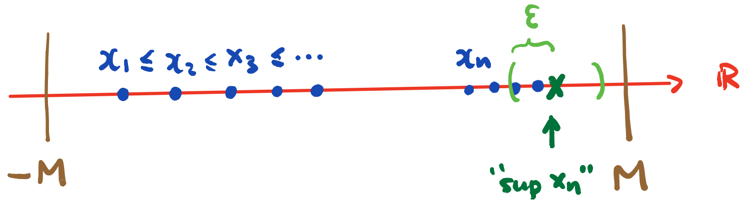
**Remark:** The theorem does NOT compute the limit.

**Non-e.g. 1)** $(x_n) = (n)$ monotone But not bdd, divergent

**Non-e.g. 2)** $(x_n) = \left(\frac{(-1)^n}{n}\right)$ NOT monotone, bdd, convergent

**Proof:** Assume $(x_n)$ is increasing & bdd.

Picture:




**Idea:** Show $\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$.

Consider the subset

$$\phi \neq S := \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R} \qquad \text{bdd} \quad (\because (x_n) \text{ bdd})$$

Completeness of $\mathbb{R}$ $\Rightarrow$ $x := \sup S \in \mathbb{R}$ exists.

**Claim:** $x = \lim(x_n)$.

**Pf of Claim:** Use def$^{\underline{n}}$ of limit. Let $\varepsilon > 0$.

Since $x = \sup S$, $x - \varepsilon$ is <u>NOT</u> an upper bd of $S$.

$\Rightarrow \exists\, K \in \mathbb{N}$ s.t. $x - \varepsilon < x_K \in S$

Since $(x_n)$ is <u>increasing</u>, we have

$$x - \varepsilon < x_K \leq x_{K+1} \leq x_{K+2} \leq \cdots \leq x_n \quad \forall n \geq K \qquad \text{——— (1)}$$

Since $x = \sup S$ is an <u>upper bd</u> of $S$,

$$x_n \leq x < x + \varepsilon \qquad \forall n \in \mathbb{N} \qquad \text{——— (2)}$$

Combining (1) & (2), $\forall n \geq K$,

$$x - \varepsilon < x_n < x + \varepsilon \qquad \text{———} \ \square$$

**Remark:** MCT is a very powerful tool to show convergence.

"Strategy"
- Step 1: use MCT to show $\lim(x_n) =: x$ exists.
- Step 2: use other ways (e.g. limit thm) to evaluate $x$.

<u>Example 1</u> : (Harmonic series)

Let $h_n := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ , $n \in \mathbb{N}$.

So, $h_1 = 1$, $h_2 = \frac{3}{2}$, $h_3 = \frac{11}{6}$, ......

Show that $(h_n)$ is divergent.

<u>Pf</u>: By MCT, $(h_n)$ divergent $\underset{(\Rightarrow)}{\Leftrightarrow}$ $(h_n)$ unbdd ($\because$ $(h_n)$ increasing)

<u>Claim</u>: $(h_n)$ is unbdd (above)

<u>Observation</u>:
$$h_1 = 1$$
$$h_2 = 1 + \frac{1}{2}$$
$$h_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}}$$
$$\vdots$$
$$h_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}}$$

More generally, if $n = 2^m$, $m \in \mathbb{N}$,
$$h_{2^m} > 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{m\text{-terms}} = 1 + \frac{m}{2}$$

unbdd as $m \to \infty$

So, $(h_n)$ is unbdd, hence divergent.

$\square$

<u>Remark</u>: MCT is particularly useful to study $(x_n)$ which are defined "recursively".

**Example 2:** (Very important)

Let $(y_n)$ be a seq. s.t. $y_1 := 1$ and

recursive formula : (#) $y_{n+1} := \frac{1}{4}(2y_n + 3)$ $\forall n \in \mathbb{N}$

Show that $\lim(y_n) = 3/2$.

**Proof:** [Observe: $y_1 = 1$, $y_2 = \frac{1}{4}(2\cdot 1 + 3) = \frac{5}{4}$, $y_3 = \frac{1}{4}(2\cdot\frac{5}{4} + 3) = \frac{11}{8}$ ...]

Claim 1: $(y_n)$ is increasing ✓ (i.e. $y_{n+1} \geq y_n$ $\forall n \in \mathbb{N}$)

Pf: By M.I., when $n=1$, $y_2 = \frac{5}{4} > 1 = y_1$.

Assume $n=k$ holds, ie $y_{k+1} \geq y_k$.

when $n = k+1$,

$$y_{k+2} := \frac{1}{4}(2y_{k+1} + 3) \geq \frac{1}{4}(2y_k + 3) =: y_{k+1} \quad \square$$

Claim 2: $(y_n)$ is bdd above by 2. ✓ (i.e. $y_n \leq 2$ $\forall n \in \mathbb{N}$)

Pf: By M.I. when $n=1$, $y_1 := 1 < 2$.

Assume $y_k \leq 2$. Then,

$$y_{k+1} := \frac{1}{4}(2y_k + 3) \leq \frac{1}{4}(2\cdot 2 + 3) = \frac{7}{4} < 2 \quad \square$$

① By MCT, $\lim(y_n) =: y$ exists.

Idea: Take the limit on both sides of (#).

$$\lim\left( y_{n+1} = \frac{1}{4}(2y_n + 3) \right) \quad ②$$

$$\underbrace{\lim(y_n)}_{} \underset{=}{=} \underbrace{\lim(y_{n+1})}_{} = \underbrace{\frac{1}{4}(2\lim(y_n) + 3)}_{\text{limit thm.}} \Rightarrow y = \frac{1}{4}(2y+3)$$

i.e. $y = 3/2$

$\because$ $(y_{n+1})$ is a 1-tail of $(y_n)$

$(y_{n+1}) = (y_2\ y_3\ y_4 \cdots y_n \cdots)$

$\square$

**Example 3:** Let $(s_n)$ be a seq. s.t. $s_1 = 2$, and

$(\#)$ : $s_{n+1} := \frac{1}{2}\left(s_n + \frac{2}{s_n}\right) \quad \forall n \in \mathbb{N}$

Show that $\lim(s_n) = \sqrt{2}$.

**Proof:** Claim 1: $(\overset{>0}{s_n})$ is bdd. from below by $\sqrt{2}$.

$(\#) \Rightarrow \quad \overset{x}{s_n}^2 - 2 s_{n+1} \overset{x}{s_n} + 2 = 0$ is a quadratic eq$^n$ in $s_n$

which has a real root (namely $s_n$)

$\Rightarrow \quad 4 s_{n+1}^2 - 4 \cdot 2 \geq 0$

$\Rightarrow \quad s_{n+1} \geq \sqrt{2}$. $\square$

Claim 2: $(s_n)$ decreasing. (ie $s_{n+1} \leq s_n \quad \forall n \in \mathbb{N}$).

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{2}{s_n}\right) = \frac{\overbrace{s_n^2 - 2}^{\geq 0 \text{ by Claim 1}}}{2 s_n} \geq 0$$

$\square$

By MCT, $\lim(s_n) =: s$ exists.

Take limit on both sides of $(\#)$. we get

$$s = \frac{1}{2}\left(s + \frac{2}{s}\right)$$

$$\Rightarrow \quad s = \sqrt{2}$$

$\square$

Remark: Since increasing (decreasing) seq. is automatically bdd below (above).

MCT: $\begin{cases} \text{increasing + bdd above} \Rightarrow \text{conv.} \\ \text{decreasing + bdd below} \Rightarrow \text{conv.} \end{cases}$

## Subsequence : (§ 3.4) difficult

**Def$^{\underline{n}}$:** Given a seq. $X = (x_n)_{n\in\mathbb{N}}$, suppose we have a strictly increasing seq. of natural numbers $n_k \in \mathbb{N}$:

$$n_1 < n_2 < n_3 < n_4 < \cdots < n_k < \cdots$$

Then,

k-th term in $X'$ = $n_k$-th term in $X$

$$X' = (x_{n_k})_{k\in\mathbb{N}} = (x_{n_1} \; x_{n_2} \; x_{n_3} \; \cdots \; x_{n_k} \; \cdots \; )$$

is called a subsequence of $(x_n)$.

**Example:**

$$(x_n) := (n) = (\underset{x_1}{1} \; 2 \; 3 \; \underset{x_4}{4} \; \underset{x_5}{5} \; \cdots \; n \; \cdots)$$

index by $n \in \mathbb{N}$

chosen: $n_1 = 1$, $n_2 = 4$, $n_3 = 5$ ……

$$(x_{n_k}) = (\underset{x_{n_1}}{1} \; \underset{x_{n_2}}{4} \; \underset{x_{n_3}}{5} \; \cdots \; \cdots \; \cdots \; \cdots)$$

index by $k \in \mathbb{N}$

**Example:** $\ell$-tail of $(x_n)$ is the subseq. $(x_{k+\ell})$.

chop this off

$$(x_n) = (x_1 \; x_2 \; \cdots \; x_\ell \; x_{\ell+1} \; x_{\ell+2} \; \cdots \; \cdots)$$

$$(\underbrace{x_{k+\ell}}_{n_k}) = (x_{\ell+1} \; x_{\ell+2} \; \cdots \; \cdots \; \cdots \; \cdots)$$

**Example:** $(x_n) = ((-1)^n)$ has subseq. $(1 \cdots 1 \cdots)$ or $(-1 \cdots -1 \cdots)$.

**Remark:** Given $(x_n)$, $\exists$ many possible subseq. $(x_{n_k})$ of $(x_n)$.

**Thm:** If $(x_n) \to x$, then ANY subseq. $(x_{n_k}) \to x$.

**Proof:** Note: $n_k \geq k \quad \forall k \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $\lim(x_n) = x$, $\exists K \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$.

Since $n_k \geq k \quad \forall k \in \mathbb{N}$, whenever $k \geq K$, we will have $n_k \geq k \geq K$

$$\Rightarrow \quad |x_{n_k} - x| < \varepsilon \quad \forall k \geq K.$$

$\square$